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The Continuous Orbifold of $\mathcal{N} = 2$ Minimal Model Holography

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ABSTRACT: For the $\mathcal{N} = 2$ Kazama-Suzuki models that appear in the duality with a higher spin theory on AdS_3 it is shown that the large level limit can be interpreted as a continuous orbifold of $2N$ free bosons and fermions by the group $U(N)$. In particular, we show that the subset of coset representations that correspond to the perturbative higher spin degrees of freedom are precisely described by the untwisted sector of this $U(N)$ orbifold. We furthermore identify the twisted sector ground states of the orbifold with specific coset representations, and give various pieces of evidence in favour of this identification.

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1 Introduction

Dualities between Vasiliev higher spin theories on Anti-de Sitter spacetimes [1] and conformal field theories constitute a promising way towards understanding the AdS/CFT correspondence. In particular, dualities of this type are ‘vector-like’, and hence contain considerably fewer degrees of freedom than the ‘adjoint-like’ theories appearing in the stringy AdS/CFT duality. Furthermore, they are in a sense weak-weak dualities, and may therefore be amenable to a perturbative proof (see [2] and [3] for reviews). This could then form the seed towards establishing the full AdS/CFT correspondence, at least at the tensionless point where a description in terms of a higher spin theory is expected to arise.

In order to understand the connection between higher spin theories and string theory in more detail, it is useful to study supersymmetric variants of the higher spin

CFT duality. Furthermore, it is natural to try to do so first in a low-dimensional setting where the higher spin theories are considerably simpler [4, 5], and also much is known about the stringy AdS/CFT duality, see [6] for a review. With this in mind, a duality between a family of $\mathcal{N} = 4$ supersymmetric coset CFTs in 2d, and a supersymmetric higher spin theory on AdS_3 was proposed in [7] and subsequently tested and extended in various ways [8–12]. This duality is a natural supersymmetric generalisation of the original bosonic higher spin CFT duality of [13].

A particularly interesting limit of the $\mathcal{N} = 4$ cosets arises for the case when one of the levels is taken to infinity since one may then hope to make contact with the D1-D5 system; this will be explored in detail [14]. As a preparation for this analysis, we study in this paper the large level limit of the $\mathcal{N} = 2$ Kazama-Suzuki cosets [15, 16] that occur in the higher spin duality with an $\mathcal{N} = 2$ supersymmetric higher spin theory on AdS_3 [17, 18]. More concretely, we consider the cosets

$$\frac{\mathfrak{su}(N+1)_k \oplus \mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1} \oplus \mathfrak{u}(1)_{N(N+1)(N+k+1)}} \quad (1.1)$$

in the limit where the level k is taken to infinity. We give convincing evidence that the limit theory has an interpretation as a $\text{U}(N)$ orbifold of $2N$ free fermions and bosons that both transform as $\mathbf{N} \oplus \bar{\mathbf{N}}$ under $\text{U}(N)$.¹ This is the natural generalisation of the bosonic analysis of [21], where it was shown that the cosets

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \quad (1.2)$$

admit a description in terms of an orbifold of $N - 1$ free bosons by the Lie group $\text{SU}(N)$. In each of these cases the limit is taken in the spirit of [22] (rather than say [23]), see also [24, 25] for other instances (closely related to the topic of this paper) where this kind of construction has been considered. We should also mention that this orbifold picture is the natural 2d analogue of the $\text{U}(N)$ (or $\text{O}(N)$) singlet sector of a theory of free bosons or fermions in 3d that played an important role in the higher spin CFT duality in one dimension higher [26, 27].

We shall consider the usual charge conjugation modular invariant of the coset (1.1) whose duality to the higher spin theory on AdS_3 was explored in [17, 18, 28–31]. In particular, we shall see that the part of the CFT spectrum that corresponds to the perturbative higher spin degrees of freedom

$$\mathcal{H}_{\text{pert}} = \bigoplus_{\Lambda} \mathcal{H}_{(0;\Lambda)} \otimes \overline{\mathcal{H}}_{(0;\Lambda^*)} \quad (1.3)$$

can be identified, for $k \rightarrow \infty$, with the subspace of the free field theory of $2N$ bosons and fermions that are singlets with respect to $\text{U}(N)$, i.e., with the untwisted

¹The idea that the limit theory has such an interpretation was already mentioned in [19], following on from the analysis of [20], where this was shown explicitly for $N = 1$.

sector of the continuous orbifold. The remaining coset primaries, i.e., those of the form $(\Lambda_+; \Lambda_-)$ with $\Lambda_+ \neq 0$, can then be interpreted in terms of the various twisted sectors of the continuous orbifold. In fact, as is familiar from usual orbifolds, the untwisted sector is not modular invariant by itself, and the twisted sectors are required in order to restore modular invariance. For the case at hand where we have supersymmetry, the identification of the different coset primaries with the twisted sectors can be worked out in detail, and a number of non-trivial consistency checks can be performed. In particular, we have compared the conformal dimension of the twisted sector ground states with that calculated from the coset viewpoint; we have also determined the fermionic excitation spectrum directly from the coset perspective.

The paper is organised as follows. In section 2 we introduce our conventions and review briefly the relevant Kazama-Suzuki coset models as well as some of their low-lying representations. In section 3 we identify the subsector of perturbative states (1.3) with the untwisted sector of the continuous orbifold, i.e., with the subspace of the free field theory that consists of the singlets under the $U(N)$ action. In particular, we show in sections 3.1 and 3.2 that the partition functions of the two descriptions agree. Section 4 is dedicated to the analysis of the twisted sectors. We identify all twisted sector ground states with coset primaries, see eq. (4.3), and show that the conformal dimensions agree. Furthermore, we compare their fermionic excitation spectrum (see section 4.2), as well as the structure of their BPS descendants (see section 4.3), and find beautiful agreement. Section 5 contains our conclusions as well as a brief outlook. We have relegated some background information and a few detailed computations to three appendices.

2 The $\mathcal{N} = 2$ Kazama-Suzuki coset model

Let us begin by introducing our conventions for the $\mathcal{N} = 2$ superconformal field theories that appear in the duality to the $\mathcal{N} = 2$ supersymmetric higher spin theory on AdS_3 [17]. The relevant cosets [15, 16] are (see [18] for our conventions)

$$\frac{\mathfrak{su}(N+1)_{k+N+1}^{(1)}}{\mathfrak{su}(N)_{k+N+1}^{(1)} \oplus \mathfrak{u}(1)_\kappa^{(1)}} \cong \frac{\mathfrak{su}(N+1)_k \oplus \mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1} \oplus \mathfrak{u}(1)_\kappa}, \quad (2.1)$$

where the second description is in terms of the bosonic affine algebras. Here the level of the $\mathfrak{u}(1)$ factor equals $\kappa = N(N+1)(N+k+1)$, and the central charge is

$$c = (N-1) + \frac{kN(N+2)}{k+N+1} - \frac{(k+1)(N^2-1)}{k+N+1} = \frac{3kN}{k+N+1}. \quad (2.2)$$

The subgroup of the denominator $SU(N) \times U(1)$ is ‘embedded’ into $SU(N+1)$ via the $(N\text{-to-one})$ mapping

$$(v, w) \mapsto \begin{pmatrix} \bar{w}v & 0 \\ 0 & w^N \end{pmatrix}, \quad (2.3)$$

where $w \in \text{U}(1)$ is a phase, while $v \in \text{SU}(N)$ is an $N \times N$ matrix. Similarly, the ‘embedding’ into $\text{SO}(N, N)$ (whose complexified Lie algebra agrees with the complexification of $\mathfrak{so}(2N)$) is defined by

$$(v, w) \mapsto \begin{pmatrix} \bar{w}^{N+1}v & 0 \\ 0 & w^{N+1}\bar{v} \end{pmatrix}, \quad (2.4)$$

see [18] for more details. Our conventions are chosen so that the free fermions and bosons have $\text{U}(1)$ charge $\pm(N+1)$.

The representations of the coset are labelled by $(\Lambda_+; \Lambda_-, \ell)$, where Λ_+ is an integrable weight of $\mathfrak{su}(N+1)_k$, Λ_- an integrable weight of $\mathfrak{su}(N)_{k+1}$, while ℓ denotes the $\mathfrak{u}(1)$ charge.² The selection rule is

$$\frac{|\Lambda_+|}{N+1} - \frac{|\Lambda_-|}{N} - \frac{\ell}{N(N+1)} \in \mathbb{Z}, \quad (2.5)$$

where $|\Lambda| = \sum_j j\Lambda_j$, and we have the field identification

$$(\Lambda_+; \Lambda_-, \ell) \cong (J^{(N+1)} \Lambda_+; J^{(N)} \Lambda_-, \ell - (k + N + 1)), \quad (2.6)$$

where J denotes the usual outer automorphism, i.e., it maps (for the case of $\mathfrak{su}(N+1)$)

$$\Lambda = [\Lambda_0; \Lambda_1, \dots, \Lambda_N] \mapsto J^{(N+1)} \Lambda = [\Lambda_N; \Lambda_0, \Lambda_1, \dots, \Lambda_{N-1}]. \quad (2.7)$$

Since the field identification acts simultaneously on a weight in $\mathfrak{su}(N+1)$ and $\mathfrak{su}(N)$, it has order $N(N+1)$; this then ties together with the fact that the $\mathfrak{u}(1)$ charge ℓ is defined modulo $\kappa = N(N+1)(N+k+1)$.

The conformal dimension of the representation $(\Lambda_+; \Lambda_-, \ell)$ equals

$$h(\Lambda_+; \Lambda_-, \ell) = \frac{C^{(N+1)}(\Lambda_+)}{N+k+1} - \frac{C^{(N)}(\Lambda_-)}{N+k+1} - \frac{\ell^2}{2N(N+1)(N+k+1)} + n, \quad (2.8)$$

where n is a half-integer, describing the ‘level’ at which (Λ_-, ℓ) appears in the representation Λ_+ , and $C^{(N)}(\Lambda)$ is the quadratic Casimir of the $\mathfrak{su}(N)$ weight Λ . Finally, the $\text{U}(1)$ charge (with respect to the $\text{U}(1)$ generator of the superconformal $\mathcal{N} = 2$ algebra) equals

$$q(\Lambda_+; \Lambda_-, \ell) = \frac{\ell}{N+k+1} + s, \quad (2.9)$$

where $s \in \mathbb{Z}$ denotes the charge contribution of the descendants. For example, the representation

$$(\mathfrak{f}; 0, N) : \quad h = \frac{N}{2(N+k+1)}, \quad q = \frac{N}{N+k+1}, \quad (2.10)$$

²Strictly speaking, the coset representations are also labelled by the representation of $\mathfrak{so}(2N)_1$. In this paper we shall only consider the NS sector of the free fermions, i.e., we shall take the $\mathfrak{so}(2N)_1$ representation to be either the vacuum or the vector representation.

where \mathbf{f} denotes the fundamental representation of $\mathfrak{su}(N+1)$, describes a chiral primary, as does

$$h(0; \mathbf{f}, -(N+1)) = \frac{1}{2} - \frac{(N^2-1)}{2N(N+k+1)} - \frac{(N+1)}{2N(N+k+1)} = \frac{k}{2(N+k+1)}, \quad (2.11)$$

for which the $U(1)$ charge equals

$$q(0; \mathbf{f}, -(N+1)) = \frac{-(N+1)}{N+k+1} + 1 = \frac{k}{N+k+1}. \quad (2.12)$$

Here the additional terms in (2.11) and (2.12) appear because for $(0; \mathbf{f}, -(N+1))$ the representation of the denominator arises only at the first excited level.

3 The continuous orbifold: the untwisted sector

We are interested in taking the $k \rightarrow \infty$ limit of these cosets. For the case $N=1$ with $c=3$, this was worked out in some detail in [20], where it was shown that the resulting theory can be interpreted in terms of a continuous $U(1)$ orbifold. Here we want to extend the discussion to general N . The idea that the limit theory may be interpreted in terms of a $U(N)$ orbifold was already sketched in [19]; in the following, we shall pursue a somewhat different approach and be much more explicit.

The discussion of [18] as well as the analogous analysis in [21] suggests that the underlying free theory consists of $2N$ free bosons and free fermions that transform as

$$\mathbf{N}_{-(N+1)} \oplus \bar{\mathbf{N}}_{N+1} \quad (3.1)$$

with respect to $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$ in the denominator. The relevant orbifold group is therefore $SU(N) \times U(1)$, or equivalently $U(N)$,³ where the group acts simultaneously on both left- and right-movers.

One reason in favour of this idea is that the central charge approximates in this limit

$$c = \frac{3kN}{k+N+1} \cong 3N, \quad (3.2)$$

in agreement with a description in terms of $2N$ free bosons and fermions. Furthermore, the ground states of the representations $(0; \mathbf{f}, -(N+1))$ and $(0; \bar{\mathbf{f}}, (N+1))$ can be identified with the $\mathbf{N} + \bar{\mathbf{N}}$ free fermions since their conformal dimension and $\mathfrak{u}(1)$ charge become in this limit

$$h(0; \mathbf{f}, -(N+1)) = h(0; \bar{\mathbf{f}}, (N+1)) = \frac{1}{2}, \quad (3.3)$$

³The discrete subgroup of $SU(N) \times U(1)$ that needs to be factored out to obtain $U(N)$ acts trivially.

as well as

$$q(0; \mathbf{f}, -(N+1)) = +1, \quad q(0; \bar{\mathbf{f}}, (N+1)) = -1. \quad (3.4)$$

Each of these representations has two $\mathcal{N} = 2$ descendants with $h = 1$, which can in turn be identified with the free bosons. For the actual coset partition function, left- and right-movers are grouped together, i.e., $(0; \mathbf{f}, -(N+1))$ for the left-movers appears together with $(0; \bar{\mathbf{f}}, (N+1))$ for the right-movers, etc., and this is precisely what the $U(N)$ singlet condition achieves.

Concretely, we therefore claim that the untwisted sector of the $U(N)$ orbifold of $2N$ free bosons and fermions, transforming as in (3.1), corresponds to the subsector of the coset theory

$$\mathcal{H}_0 = \bigoplus_{\Lambda, u} \mathcal{H}_{(0; \Lambda, u)} \otimes \bar{\mathcal{H}}_{(0; \Lambda^*, -u)} \quad (3.5)$$

in the limit $k \rightarrow \infty$. Here the sum runs over all representations Λ that appear in finite tensor powers of the fundamental or anti-fundamental representation of $\mathfrak{su}(N)$ — in the limit $k \rightarrow \infty$, the k -dependent bound on the integrable $\mathfrak{su}(N)_{k+1}$ representations disappears — and Λ^* denotes the representation conjugate to Λ . Furthermore, u must satisfy the selection rule that $(N+1)|\Lambda| + u = 0 \pmod{N(N+1)}$.

In the following we will give strong evidence in favour of this claim by showing that the partition functions agree. In section 4 we shall then also explain how the twisted sectors of the continuous orbifold can be understood from the coset viewpoint.

3.1 The partition function from the coset

We want to show that the spectrum of the untwisted sector of the $U(N)$ orbifold coincides with eq. (3.5) by comparing partition functions. In order to do so, we need to understand the character of the coset representations $(0; \Lambda, u)$ in the limit $k \rightarrow \infty$. For large k , the character of an affine representation Λ of $\mathfrak{su}(N)_k$ is given by

$$\text{ch}_\Lambda^{N,k}(v; q) = \frac{q^{h_\Lambda^{N,k}} [\text{ch}_\Lambda^N(v) + \mathcal{O}(q^{k - \sum_i \Lambda_i + 1})]}{\prod_{n=1}^\infty [(1 - q^n)^{N-1} \prod_{i \neq j} (1 - v_i \bar{v}_j q^n)]}. \quad (3.6)$$

Here v_i are the eigenvalues of $v \in \text{SU}(N)$, $\text{ch}_\Lambda^N(v)$ is the character of Λ restricted to the zero mode subalgebra $\mathfrak{su}(N)$, Λ_i are the Dynkin labels of Λ , and we define

$$h_\Lambda^{N,k} = \frac{C^{(N)}(\Lambda)}{N+k}, \quad (3.7)$$

where $C^{(N)}(\Lambda)$ is, as before, the quadratic Casimir of Λ . For example, the vacuum character $\text{ch}_0^{N+1,k}(v, w; q)$ of $\mathfrak{su}(N+1)_k$ with $v \in \text{SU}(N)$ and $w \in U(1)$ embedded into $\text{SU}(N+1)$ as in (2.3) equals

$$\text{ch}_0^{N+1,k} = \frac{1 + \mathcal{O}(q^{k+1})}{\prod_{n=1}^\infty [(1 - q^n)^N \prod_{i \neq j} (1 - v_i \bar{v}_j q^n) \prod_{i=1}^N [(1 - \bar{w}^{N+1} v_i q^n)(1 - w^{N+1} \bar{v}_i q^n)]]}. \quad (3.8)$$

Moreover, the representations of the $\mathfrak{so}(2N)_1$ factor in the numerator are the vacuum and vector representation, as well as either of the two spinor representations. In terms of the free fermions (that are equivalent to $\mathfrak{so}(2N)_1$), the former two correspond to the NS sector, while the latter are accounted for in terms of the R sector. In the following we shall concentrate on the NS sector⁴ for which the contribution of the $2N$ free fermions equals

$$\theta(v, w; q) = \prod_{n=1}^{\infty} \prod_{i=1}^N (1 + \bar{w}^{N+1} v_i q^{n-\frac{1}{2}}) (1 + w^{N+1} \bar{v}_i q^{n-\frac{1}{2}}) . \quad (3.9)$$

The characters of the denominator, on the other hand, are given in that limit by

$$\text{ch}_{\Lambda, u}^{N, k+1}(v, w; q) = \frac{q^{h_{\Lambda}^{N, k+1} + \frac{u^2}{2\kappa}} (w^u + \mathcal{O}(q^{\frac{\kappa}{2} - |u|})) (\text{ch}_{\Lambda}^N(v) + \mathcal{O}(q^{k - \sum_i \Lambda_i + 2}))}{\prod_{n=1}^{\infty} [(1 - q^n)^N \prod_{i \neq j} (1 - v_i \bar{v}_j q^n)]} . \quad (3.10)$$

The coset character associated to $(0; \Lambda, u)$ is then given by the branching function $b_{0; \Lambda, u}^{N, k}(q)$, which is defined by

$$\text{ch}_0^{N+1, k}(v, w; q) \theta(v, w; q) = \sum_{\Lambda, u} b_{0; \Lambda, u}^{N, k}(q) \text{ch}_{\Lambda, u}^{N, k+1}(v, w; q) . \quad (3.11)$$

Combining the explicit expressions given above, the branching functions take the form (see also [18])

$$b_{0; \Lambda, u}^{N, k}(q) = q^{-h_{\Lambda}^{N, k+1} - \frac{u^2}{2\kappa}} \left[a_{0; \Lambda, u}^N(q) + \mathcal{O}(q^{k - \sum_i \Lambda_i + 2}) + \mathcal{O}(q^{\frac{\kappa}{2} - |u|}) \right] , \quad (3.12)$$

where $a_{0; \Lambda, u}^N(q)$ is the multiplicity of $w^u \text{ch}_{\Lambda}^N(v)$ in

$$\sum_{\Lambda, u} a_{0; \Lambda, u}^N(q) w^u \text{ch}_{\Lambda}^N(v) = \prod_{n=1}^{\infty} \prod_{i=1}^N \frac{(1 + \bar{w}^{N+1} v_i q^{n-\frac{1}{2}}) (1 + w^{N+1} \bar{v}_i q^{n-\frac{1}{2}})}{(1 - \bar{w}^{N+1} v_i q^n) (1 - w^{N+1} \bar{v}_i q^n)} . \quad (3.13)$$

It therefore follows that the partition function \mathcal{Z}_0 of (3.5) equals for $k \rightarrow \infty$

$$\mathcal{Z}_0 = \lim_{k \rightarrow \infty} (q\bar{q})^{-\frac{c}{24}} \sum_{\Lambda, u} |b_{0; \Lambda, u}^{N, k}(q)|^2 = (q\bar{q})^{-\frac{N}{8}} \sum_{\Lambda, u} |a_{0; \Lambda, u}^N(q)|^2 , \quad (3.14)$$

where we sum over all finite Young diagrams Λ of at least $N - 1$ rows, and u must be of the form $u = (N + 1)(-|\Lambda| + nN)$ with $n \in \mathbb{Z}$. In the second equality, we have used that since Λ and u are finite (and do not grow with k), the prefactor in eq. (3.12), $h_{\Lambda}^{N, k+1} + \frac{u^2}{2\kappa}$, vanishes in the limit, and the higher order terms in the bracket become irrelevant.

⁴In the duality to the higher spin theory on AdS_3 only the NS-NS sector plays a role since the conformal dimension of the RR sector states is proportional to the central charge, see the discussion in [32].

3.2 Comparison with the untwisted orbifold sector

We shall now compare this result to the $U(N)$ orbifold of $2N$ free fermions and bosons that transform as $\mathbf{N} \oplus \bar{\mathbf{N}}$ of $U(N)$, cf., eq. (3.1). Labelling again the elements of $U(N)$ in terms of $SU(N) \times U(1)$ via the ‘embedding’

$$\iota: (v, w) \mapsto w^{-(N+1)} \cdot v = \bar{w}^{(N+1)} \cdot v, \quad (3.15)$$

the partition function with the insertion of these group elements takes the form

$$\iota(v, w) \cdot \mathcal{Z}_{\text{free}} = (q\bar{q})^{-\frac{N}{8}} \prod_{n=1}^{\infty} \prod_{i=1}^N \frac{|1 + \bar{w}^{(N+1)} v_i q^{n-\frac{1}{2}}|^2 |1 + w^{N+1} \bar{v}_i q^{n-\frac{1}{2}}|^2}{|1 - \bar{w}^{(N+1)} v_i q^n|^2 |1 - w^{N+1} \bar{v}_i q^n|^2}, \quad (3.16)$$

where we have used that the central charge equals $c = 3N$. The untwisted sector of this orbifold theory consists of the states that are $U(N)$ invariant. Put differently, the untwisted sector is therefore the multiplicity space of the trivial representation of $U(N)$ acting on the free theory with partition function $\mathcal{Z}_{\text{free}}$. Since (3.16) is, up to the prefactor, just the charge-conjugate square of the coset numerator character (3.13), this amounts to finding the trivial representation in

$$(0; \Lambda_1, u_1) \otimes (0; \Lambda_2, u_2) \quad (3.17)$$

for some representations (Λ_i, u_i) ($i = 1, 2$) of $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$, where the first factor corresponds to the left-movers and the second one to the right-movers. This tensor product contains the trivial representation if and only if $\Lambda_1 = \Lambda_2^*$ and $u_1 = -u_2$, where Λ_2^* is the representation conjugate to Λ_2 , and it always does so with multiplicity one. Thus we conclude that the partition function of the untwisted sector equals

$$\mathcal{Z}_U = (q\bar{q})^{-\frac{N}{8}} \sum_{\Lambda, u} |a_{0; \Lambda, u}^N(q)|^2, \quad (3.18)$$

matching precisely (3.14). This yields convincing evidence that the coset subsector of states $(0; \Lambda, u)$ can indeed be described by the untwisted sector of the $U(N)$ orbifold introduced above.

4 Twisted sectors

The remaining states, i.e., those with $\Lambda_+ \neq 0$, should then arise from the twisted sector of the continuous orbifold. In the following we shall be able to make this correspondence rather concrete. The main reason why we can be much more explicit (see eq. (4.5) below) than in the corresponding bosonic analysis of [21] is that the $\mathcal{N} = 2$ superconformal symmetry is quite restrictive and in particular implies that the ground state energy of the twisted sectors is linear in the twist.

To begin with, let us briefly review the basic logic of the continuous orbifold approach of [21]. As was explained there, continuous compact groups (such as $U(N)$) behave in many respects like finite groups, and one may therefore believe that an orbifold by a continuous compact group can be constructed essentially as in the familiar finite case. In particular, the untwisted sector just consists of the invariant states of the original theory, while the twisted sectors are labelled by the conjugacy classes of the orbifold group. Finally, in each such twisted sector, only the states that are invariant with respect to the centraliser of the twist element survive.

For the case of $U(N)$, the conjugacy classes are labelled by the elements in the Cartan torus $U(1)^N$ modulo the action of the Weyl group, i.e., the permutation group S_N . Furthermore, the centraliser of a generic element of the Cartan torus is again just the Cartan torus itself, i.e., the orbifold projection in the twisted sector will just guarantee that the partition function is invariant under the T -transformation, $\tau \mapsto \tau + 1$.

Let us parametrise the elements of the Cartan torus by the diagonal matrices

$$\text{diag}(e^{2\pi i \alpha_1}, \dots, e^{2\pi i \alpha_N}) , \quad -\frac{1}{2} < \alpha_i \leq \frac{1}{2} \quad (i = 1, \dots, N) . \quad (4.1)$$

Since the Weyl group permutes these entries, the conjugacy classes (and thus the twisted sectors) can actually be labelled by

$$\alpha = [\alpha_1, \dots, \alpha_N] , \quad (4.2)$$

where now, in addition, $\alpha_i \leq \alpha_j$ for $i < j$. In this section, we will argue that the ground state of the sector with twist α can be identified, in the limit $k \rightarrow \infty$, with the coset representative

$$\left(\Lambda_+(\alpha); \Lambda_-(\alpha), u(\alpha) \right) , \quad (4.3)$$

where $m \in \{0, \dots, N\}$ is chosen such that

$$\alpha_i \leq 0 \text{ for } i \leq m \quad \text{and} \quad \alpha_i \geq 0 \text{ for } i > m , \quad (4.4)$$

and we define

$$\begin{aligned} \Lambda_+(\alpha) = [k(\alpha_2 - \alpha_1), \dots, k(\alpha_m - \alpha_{m-1}), -k\alpha_m, \\ k\alpha_{m+1}, k(\alpha_{m+2} - \alpha_{m+1}), \dots, k(\alpha_N - \alpha_{N-1})] , \end{aligned} \quad (4.5)$$

$$\Lambda_-(\alpha) = [k(\alpha_2 - \alpha_1), \dots, k(\alpha_N - \alpha_{N-1})] , \quad u(\alpha) = k \sum_{i=1}^N \alpha_i , \quad (4.6)$$

where each entry of the weights is projected onto the integer part (and we also adjust $u(\alpha)$ correspondingly). These weights are then allowed at level k since we have

$$\sum_{j=1}^{N+1} [\Lambda_+(\alpha)]_j = \sum_{j=1}^N [\Lambda_-(\alpha)]_j = k(\alpha_N - \alpha_1) \leq k . \quad (4.7)$$

One also easily checks that (4.3) satisfies the selection rule (2.5). Conversely, for every coset primary $(\Lambda_+; \Lambda_-, u)$, we can write, after a suitable field redefinition if necessary, $\Lambda_+ \equiv \Lambda_+(\alpha)$ for some α of the form (4.2) with $-\frac{1}{2} < \alpha_i \leq \frac{1}{2}$ and $\alpha_i \leq \alpha_j$ for $i < j$; indeed, the corresponding α may be taken to be

$$\alpha = \frac{1}{k} \left[-\sum_{i=1}^m \Lambda_i, -\sum_{i=2}^m \Lambda_i, \dots, -\Lambda_m, \Lambda_{m+1}, \sum_{i=m+1}^{m+2} \Lambda_i, \dots, \sum_{i=m+1}^N \Lambda_i \right], \quad (4.8)$$

where we choose m such that

$$\sum_{i=1}^m \Lambda_i < \frac{k}{2}, \quad \text{and} \quad \sum_{i=m+1}^N \Lambda_i \leq \frac{k}{2}. \quad (4.9)$$

We will give three main pieces of evidence for this identification: we will show in section 4.1 that the conformal dimension of the coset primary (4.3) agrees with the ground state energy of the α -twisted state; we will confirm that the fermionic excitation spectrum of the coset primary has the expected form (see section 4.2); and we shall show in section 4.3 that the twisted sector has BPS descendants precisely as suggested by the orbifold picture.

4.1 Conformal dimension

In the α -twisted sector the free fermions and bosons are simultaneously twisted (as they transform in the same representation of $U(N)$, see eq. (3.1) above). As a consequence, the ground state energy of the α -twisted sector should simply be

$$h(\alpha) = \frac{1}{2} \sum_{i=1}^N |\alpha_i|. \quad (4.10)$$

(For the convenience of the reader we have outlined the calculation of the twisted sector ground state energy in appendix A, see in particular eq. (A.14).) We therefore need to show that the conformal dimension of (4.3) agrees with (4.10).

In order to determine the conformal dimension of (4.3), we use (2.8) and note that the quadratic Casimir of a weight Λ of $\mathfrak{su}(N)$ is given by

$$C^{(N)}(\Lambda) = \sum_{i < j} \Lambda_i \Lambda_j \frac{i(N-j)}{N} + \frac{1}{2} \sum_j \Lambda_j^2 \frac{j(N-j)}{N} + \sum_j \Lambda_j \frac{j(N-j)}{2}. \quad (4.11)$$

The key step of the computation is to calculate the difference of the Casimirs, which turns out to equal

$$\Delta C = C^{(N+1)}(\Lambda_+(\alpha)) - C^{(N)}(\Lambda_-(\alpha)) = \frac{\left(k \sum_{i=1}^N \alpha_i\right)^2}{2N(N+1)} + \frac{k}{2} \left(-\sum_{i=1}^m \alpha_i + \sum_{i=m+1}^N \alpha_i \right). \quad (4.12)$$

It then follows that the conformal dimension is indeed given by

$$\begin{aligned} h\left(\Lambda_+(\alpha); \Lambda_-(\alpha), u(\alpha)\right) &= \frac{\Delta C}{N+k+1} - \frac{u(\alpha)^2}{2N(N+1)(N+k+1)} \\ &= \frac{k}{2(N+k+1)} \left(-\sum_{i=1}^m \alpha_i + \sum_{i=m+1}^N \alpha_i \right) \cong \frac{1}{2} \sum_{i=1}^N |\alpha_i| \end{aligned} \quad (4.13)$$

in the limit $k \rightarrow \infty$. Here we have used that the excitation number n in (2.8) vanishes because the representation $\Lambda_-(\alpha)$ appears in the branching of $\Lambda_+(\alpha)$ from $\mathfrak{su}(N+1)$ to $\mathfrak{su}(N)$, as follows from the discussion in appendix B.

We should also mention that the $U(1)$ charge of the coset primary equals

$$q\left(\Lambda_+(\alpha); \Lambda_-(\alpha), u(\alpha)\right) = \frac{u(\alpha)}{N+k+1} \cong \sum_{i=1}^N \alpha_i, \quad (4.14)$$

which also agrees with what one expects based on the twisted sector analysis. Note that the ground state is a chiral primary if all twists are positive, and an anti-chiral primary if all twists are negative; we shall come back to a more detailed analysis of the BPS states in the twisted sectors in section 4.3.

4.2 The fermionic excitation spectrum

We can test the above correspondence further by calculating the actual excitation spectrum of the fermions in the twisted sector. Recall that the free fermions correspond to the coset primaries $(0; \mathbf{f}, -(N+1))$ and $(0; \bar{\mathbf{f}}, (N+1))$, respectively. We can therefore determine the ‘twist’ of these fermions by evaluating the change in conformal dimension upon fusion with these fields. As a by-product of this analysis we will also be able to show that the above coset primaries are indeed ground states.

More specifically, suppose that $(\Lambda_+; \Lambda_-, u)$ is the (ground) state of a twisted sector. Then we consider the fusion products

$$(\Lambda_+; \Lambda_-, u) \otimes (0; \mathbf{f}, -(N+1)) = \bigoplus_{l=0}^{N-1} (\Lambda_+; \Lambda_-^{-(l)}, u - (N+1)), \quad (4.15)$$

where $\Lambda_-^{-(l)}$ with $l = 0, \dots, N-1$ denotes the N different weights that appear in the tensor product $\Lambda \otimes \mathbf{f}$. Similarly we define

$$(\Lambda_+; \Lambda_-, u) \otimes (0; \bar{\mathbf{f}}, (N+1)) = \bigoplus_{l=0}^{N-1} (\Lambda_+; \Lambda_-^{+(l)}, u + (N+1)), \quad (4.16)$$

where $\Lambda_-^{+(l)}$ labels the weights that appear in $\Lambda \otimes \bar{\mathbf{f}}$; a closed formula for both cases is given by

$$\Lambda_j^{\epsilon(l)} = \begin{cases} \Lambda_j + \epsilon & j = l \\ \Lambda_j - \epsilon & j = l + 1 \\ \Lambda_j & \text{otherwise.} \end{cases} \quad (4.17)$$

Here $\epsilon = \pm$, and we have assumed that all $\Lambda_j \neq 0$ so that all N fusion channels $\Lambda^{\epsilon(l)}$ are indeed allowed. (We will comment on the situation when this is not the case at the end of this subsection.)

Now the ‘twist’ of the fermionic excitations of the twisted sector state $(\Lambda_+; \Lambda_-, u)$ can be determined by calculating the difference of conformal dimension of the coset primaries that appear in (4.15) and (4.16), relative to the original state. Indeed, generically, there will be N different such twists, corresponding to the N different fusion channels in (4.17), and this ties in with the fact that there are N fundamental fermions (as well as their conjugates). One cross-check of our analysis will be that the twists of the fermions and their conjugates will be opposite, and this will indeed turn out to be the case.

In order to calculate this difference of conformal dimension we note that it follows from (2.8) that

$$\begin{aligned} \delta h^{(l)} &\equiv h\left(\Lambda_+; \Lambda_-^{\epsilon(l)}, u + \epsilon(N+1)\right) - h(\Lambda_+; \Lambda_-, u) \\ &= \frac{1}{N+k+1} \left(C^{(N)}(\Lambda_-) - C^{(N)}(\Lambda_-^{\epsilon(l)}) \right) \\ &\quad - \frac{1}{2N(N+1)(N+k+1)} \left(2\epsilon u(N+1) + (N+1)^2 \right) + n . \end{aligned} \quad (4.18)$$

The difference of Casimir operators turns out to equal

$$\begin{aligned} \delta C^{(l)} &= C^{(N)}(\Lambda_-) - C^{(N)}(\Lambda_-^{\epsilon(l)}) \\ &= -\frac{\epsilon}{N} \sum_{i=1}^{N-1} i \Lambda_i + \epsilon \sum_{j=l+1}^{N-1} \Lambda_j + \frac{1}{2N} (\epsilon N^2 - 2\epsilon N - \epsilon N + 1 - N) , \end{aligned} \quad (4.19)$$

where Λ_j are the Dynkin labels of Λ_- . Thus we find that

$$\begin{aligned} \delta h^{(l)} &= n + \frac{1}{N+k+1} \left[-\frac{\epsilon}{N} \left(\sum_{i=1}^{N-1} i \Lambda_i + u \right) + \epsilon \sum_{j=l+1}^{N-1} \Lambda_j \right] \\ &\quad + \frac{1}{2(N+k+1)} (\epsilon N - 2\epsilon - (2 + \epsilon)) . \end{aligned} \quad (4.20)$$

In the limit $k \rightarrow \infty$, the second line can be ignored (since none of the terms in the numerator can depend on k), and hence we get approximately

$$\delta h^{(l)} \cong n + \frac{\epsilon}{N+k+1} \left[\sum_{j=l+1}^{N-1} \Lambda_j - \frac{1}{N} \left(\sum_{i=1}^{N-1} i \Lambda_i + u \right) \right] . \quad (4.21)$$

Applying this formula to the state (4.3) and using (4.5) yields then

$$\delta h^{(l)} \cong n - \epsilon \alpha_{l+1} , \quad (4.22)$$

where α_{l+1} denotes the different components of the twist in (4.2). For the free fermions,⁵ the selection rule of the $\mathfrak{so}(2N)_1$ factor implies that $n = \frac{1}{2}$. Thus, the excitations of the fermions are shifted away from the untwisted NS value $\delta h = \frac{1}{2}$ by the twist α_{l+1} . Furthermore, this twist is opposite for the fermions and the anti-fermions, i.e., it is proportional to ϵ . This then agrees precisely with what should be the case for the α -twisted sector.

It is worth stressing that the derivation of (4.21) was completely general, and did, in particular, not assume any specific properties of the state $(\Lambda_+; \Lambda_-, u)$. Thus we can use it to read off the twist of *any* coset state, which therefore equals

$$\alpha_j \cong -\frac{1}{N+k+1} \left[\sum_{i=j}^{N-1} \Lambda_i - \frac{1}{N} \left(\sum_{i=1}^{N-1} i \Lambda_i + u \right) \right], \quad (4.23)$$

where the Λ_j are, as before, the Dynkin labels of Λ_- . Note that finite excitations only change the Λ_i and u by a finite amount, which can be neglected in the limit $k \rightarrow \infty$. We therefore conclude that finitely excited states live in the same twisted sector as the corresponding ground state. Again, this is what should be the case for the α -twisted sector.

Finally, we comment on the special situation for which some of the $\Lambda_j = 0$. In that case, there are actually fewer fermionic excitations since some of the l in (4.17) are not allowed. This phenomenon also has a very natural interpretation from the continuous orbifold perspective: because of eq. (4.5), $\Lambda_j = 0$ implies that $\alpha_{j+1} = \alpha_j$. Then the centraliser of the corresponding element of the Cartan torus (4.1) is *bigger* than just the Cartan torus itself, since it includes, in particular, the $SU(2)$ subgroup that rotates the two twists α_j and α_{j+1} into one another. This means that actually fewer fermionic excitations survive the orbifold projection in the twisted sector, in perfect agreement with the fact that we also have fewer coset descendants. The analysis works similarly if more than one $\Lambda_j = 0$, etc.

It remains to show that the coset states (4.3) actually correspond to the *ground states* of the α -twisted sector. For the fermionic excitations with $n = \frac{1}{2}$ this is obvious from the above (given that, by construction, each $|\alpha_j| \leq \frac{1}{2}$). The argument for the bosonic descendants (for which $n = 0$ is possible) requires more work and is spelled out in appendix C.

4.3 BPS descendants

Finally, we want to analyse the BPS descendants of the twisted sector ground states. For the case with $\mathcal{N} = 4$ superconformal symmetry, it is well known from the analysis of the symmetric orbifold, see e.g., [36], that each twisted sector of the symmetric

⁵Technically, this means we have to consider the so-called ‘even’ fusion of the associated coset fields, see [33, 34], as well as [35]. In order to analyse the bosonic descendants (that sit in the same $\mathcal{N} = 2$ representation), we then have to consider the ‘odd’ fusion rules.

orbifold contains a BPS descendant that is obtained from the twisted sector ground state upon applying all fermionic generators whose mode number is less than $1/2$. For the case at hand, i.e., the situation with $\mathcal{N} = 2$ superconformal symmetry, we expect that each twisted sector should contain *two* BPS states, one chiral primary that is obtained by applying all $q = +1$ fermionic modes whose mode number is less than $1/2$ to the twisted sector ground state; and one anti-chiral primary that is obtained by applying all $q = -1$ fermionic modes whose mode number is less than $1/2$. Actually, as we shall see, this expectation is borne out; quite surprisingly, the relevant chiral and anti-chiral states remain BPS even at finite N and k .

To be more specific, let us consider the twisted sector ground state defined in eq. (4.3). In order to obtain the chiral primary descendant we have to apply the fermionic modes associated to $(0; \mathbf{f}, -(N+1))$ whose mode numbers are less than $1/2$. Thus we should consider the descendant where we add a box to each of the first m rows, i.e., the coset primary

$$\left(\Lambda_+(\alpha); \Lambda_-(\alpha)^{(\text{BPS})}, u(\alpha)^{(\text{BPS})} \right), \quad (4.24)$$

where, for $m \geq 1$,

$$\Lambda_-(\alpha)^{(\text{BPS})} = \left[k(\alpha_2 - \alpha_1), \dots, k(\alpha_{m+1} - \alpha_m) + 1, \dots, k(\alpha_N - \alpha_{N-1}) \right], \quad (4.25)$$

and

$$u(\alpha)^{(\text{BPS})} = k \sum_{i=1}^N \alpha_i - m(N+1). \quad (4.26)$$

We now claim that this defines a chiral primary operator, even for finite N and k .⁶ Similarly, the anti-chiral primary is obtained by applying the fermionic modes associated to $(0; \bar{\mathbf{f}}, (N+1))$ whose mode numbers are less than $1/2$, i.e., by removing a box in each of the rows $m+1, \dots, N$. The corresponding anti-chiral primary is then

$$\left(\Lambda_+(\alpha); \Lambda_-(\alpha)^{(\overline{\text{BPS}})}, u(\alpha)^{(\overline{\text{BPS}})} \right), \quad (4.27)$$

where, for $m < N$,

$$\Lambda_-(\alpha)^{(\overline{\text{BPS}})} = \left[k(\alpha_2 - \alpha_1), \dots, k(\alpha_{m+1} - \alpha_m) + 1, \dots, k(\alpha_N - \alpha_{N-1}) \right] = \Lambda_-(\alpha)^{(\text{BPS})}, \quad (4.28)$$

but now

$$u(\alpha)^{(\overline{\text{BPS}})} = k \sum_{i=1}^N \alpha_i + (N-m)(N+1). \quad (4.29)$$

⁶One way to see this is to note that, up to a field identification, this coset primary satisfies $\Lambda_- = P\Lambda_+$, where P is the restriction to the first $N-1$ Dynkin labels. We thank Stefan Fredenhagen for pointing this out to us.

Note that both states satisfy the selection rule (2.5) because

$$|\Lambda_+(\alpha)| = -k \sum_{i=1}^N \alpha_i + (N+1)k\alpha_N \quad (4.30)$$

and

$$|\Lambda_-(\alpha)^{(\text{BPS})}| = |\Lambda_-(\alpha)^{(\overline{\text{BPS}})}| = -k \sum_{i=1}^N \alpha_i + Nk\alpha_N + m. \quad (4.31)$$

To show that these states are indeed chiral and anti-chiral primaries, we again first compute the difference of the Casimirs; using the result from (4.12) we obtain

$$\begin{aligned} \Delta C &= C^{(N+1)}(\Lambda_+(\alpha)) - C^{(N)}(\Lambda_-(\alpha)^{(\text{BPS})}) \\ &= C^{(N+1)}(\Lambda_+(\alpha)) - C^{(N)}(\Lambda_-(\alpha)) + k \sum_{i=1}^m \alpha_i - \frac{mk}{N} \sum_{i=1}^N \alpha_i - \frac{N+1}{2N} m(N-m) \\ &= \frac{[u(\alpha)^{(\text{BPS})}]^2}{2N(N+1)} + \frac{1}{2} u(\alpha)^{(\text{BPS})}. \end{aligned} \quad (4.32)$$

Eqs. (2.8) and (2.9) then directly lead to

$$\begin{aligned} h\left(\Lambda_+(\alpha); \Lambda_-(\alpha)^{(\text{BPS})}, u(\alpha)^{(\text{BPS})}\right) &= \frac{u(\alpha)^{(\text{BPS})}}{2(N+k+1)} + \frac{m}{2} \\ &= \frac{1}{2} q\left(\Lambda_+(\alpha); \Lambda_-(\alpha)^{(\text{BPS})}, u(\alpha)^{(\text{BPS})}\right), \end{aligned} \quad (4.33)$$

so these states are indeed chiral primary. Similarly, using

$$u(\alpha)^{(\text{BPS})} = u(\alpha)^{(\overline{\text{BPS}})} - N(N+1) \quad (4.34)$$

we compute

$$\begin{aligned} h\left(\Lambda_+(\alpha); \Lambda_-(\alpha)^{(\overline{\text{BPS}})}, u(\alpha)^{(\overline{\text{BPS}})}\right) &= -\frac{u(\alpha)^{(\overline{\text{BPS}})}}{2(N+k+1)} + \frac{N-m}{2} \\ &= -\frac{1}{2} q\left(\Lambda_+(\alpha); \Lambda_-(\alpha)^{(\overline{\text{BPS}})}, u(\alpha)^{(\overline{\text{BPS}})}\right), \end{aligned} \quad (4.35)$$

and thus these states are anti-chiral primary as claimed.

Note that for $m = 0$, all twists are non-negative, and so by (4.13) and (4.14) already the ground state is chiral primary. Similarly, the ground state with $m = N$ is anti-chiral primary since all twists are non-positive.

5 Conclusions and outlook

In this paper we have collected evidence for the assertion that the $\mathcal{N} = 2$ $\text{SU}(N)$ Kazama-Suzuki models that occur in the duality with the higher spin theory on AdS_3

can be described, in the $k \rightarrow \infty$ limit, by a $U(N)$ orbifold of $2N$ free bosons and fermions. In particular, the subsector of the coset theory consisting of the states of the form $(0; \Lambda, u)$ — these are dual to the excitations of one complex scalar multiplet of the higher spin theory — corresponds to the untwisted sector of this orbifold, as follows from the comparison of the partition functions. We have also identified the twisted sector ground states from the coset perspective, and shown that their conformal dimension, their excitation spectrum and their BPS descendants match the orbifold prediction. In particular, the BPS states are generated from the ground states by exciting them with all fermions or antifermions whose twist has the same sign.

Our analysis was motivated by the duality [7] relating the family of Wolf space cosets with large $\mathcal{N} = 4$ superconformal symmetry to the $\mathcal{N} = 4$ superconformal higher spin theory on AdS_3 . In this case, the $k \rightarrow \infty$ limit corresponds to the situation where the radius of one of the two S^3 's in $AdS_3 \times S^3 \times S^3 \times S^1$ becomes infinite, and one may hope to make contact with string theory on $AdS_3 \times S^3 \times T^4$; this will be explored in more detail elsewhere [14].

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A Twisted sector ground state energies

In this appendix we collect together some formulae for the ground state energies of twisted fermions and bosons.

A.1 Complex free fermions

We begin with the case of free fermions twisted by α with $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$. Let us consider a pair of complex fermions that pick up eigenvalues $e^{\pm 2\pi i \alpha}$ under the twist. The relevant twining character, i.e. the character with the insertion of the eigenvalues $e^{\pm 2\pi i \alpha}$, equals then in the NS-sector

$$\chi_\alpha(\tau) = \frac{\vartheta_3(\tau, \alpha)}{\eta(\tau)}, \quad (A.1)$$

where we use the definitions

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) , \quad (\text{A.2})$$

$$\vartheta_3(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n) (1 + yq^{n-1/2}) (1 + y^{-1}q^{n-1/2}) , \quad (\text{A.3})$$

as well as $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$. To obtain the ground state energy of the twisted sector we perform an S -modular transformation, using the transformation rules

$$\eta(-\frac{1}{\tau}) = (-i\tau)^{1/2} \eta(\tau) \quad (\text{A.4})$$

$$\vartheta_3(-\frac{1}{\tau}, \frac{z}{\tau}) = (-i\tau)^{1/2} e^{i\pi z^2/\tau} \vartheta_3(\tau, z) , \quad (\text{A.5})$$

to obtain for the α -twisted partition function

$$\chi_\alpha(-\frac{1}{\tau}) = e^{i\pi\alpha^2\tau} \frac{\vartheta_3(\tau, \tau\alpha)}{\eta(\tau)} \quad (\text{A.6})$$

$$= q^{-\frac{1}{24}} e^{i\pi\alpha^2\tau} \prod_{n=1}^{\infty} (1 + e^{2\pi i\tau\alpha} q^{n-1/2}) (1 + e^{-2\pi i\tau\alpha} q^{n-1/2}) . \quad (\text{A.7})$$

Thus the ground state energy of the α -twisted sector equals

$$\Delta h_{\text{fer}} = \frac{1}{2} \alpha^2 . \quad (\text{A.8})$$

A.2 Complex free bosons and susy case

The analysis for a pair of complex bosons is essentially identical. Now the relevant twining character equals

$$\chi_\alpha(\tau) = -2 \sin(\pi\alpha) \frac{\eta(\tau)}{\vartheta_1(\tau, \alpha)} , \quad (\text{A.9})$$

where $\vartheta_1(\tau, z)$ is defined by

$$\vartheta_1(\tau, z) = -2q^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^n) (1 - yq^n) (1 - y^{-1}q^n) . \quad (\text{A.10})$$

The modular transformation behaviour of $\vartheta_1(\tau, z)$ is

$$\vartheta_1(-\frac{1}{\tau}, \frac{z}{\tau}) = -i(i\tau)^{1/2} e^{i\pi z^2/\tau} \vartheta_1(\tau, z) , \quad (\text{A.11})$$

and hence the twisted character equals

$$\chi_\alpha(-\frac{1}{\tau}) = i \frac{\sin(\pi\alpha)}{\sin(\pi\tau\alpha)} e^{-i\pi\alpha^2\tau} q^{-\frac{2}{24}} \prod_{n=1}^{\infty} \frac{1}{(1 - e^{2\pi i\alpha\tau} q^n) (1 - e^{-2\pi i\alpha\tau} q^n)} . \quad (\text{A.12})$$

For $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$ we read off from the leading $q \rightarrow 0$ behaviour that

$$\Delta h_{\text{bos}} = \frac{1}{2}|\alpha| - \frac{1}{2}\alpha^2 . \quad (\text{A.13})$$

Note that for a supersymmetric theory, i.e., for a theory where both bosons and fermions are twisted by the same amount, the total ground state energy is then

$$\Delta h_{\text{tot}} = \Delta h_{\text{bos}} + \Delta h_{\text{fer}} = \frac{|\alpha|}{2} , \quad (\text{A.14})$$

which is indeed linear in $|\alpha|$.

B Branching rules

In this appendix we explain the branching rules of $\mathfrak{su}(N+1) \supset \mathfrak{su}(N)$. They were first derived by Weyl [37] in terms of $\mathfrak{u}(N)$ tensors (see, e.g., [38] and [39] for more modern and general treatments).

Let $\Lambda = [\Lambda_1, \dots, \Lambda_N]$ be a highest weight of $\mathfrak{su}(N+1)$. The procedure can be divided into three steps:

1. Interpret Λ as a highest weight of $\mathfrak{u}(N+1)$ rather than $\mathfrak{su}(N+1)$.
2. Let r_i denote the number of boxes in the i^{th} row of the Young diagram associated with Λ ,

$$r_i = \sum_{j=i}^N \Lambda_j . \quad (\text{B.1})$$

Then under the branching $\mathfrak{u}(N+1) \supset \mathfrak{u}(N)$ we have the decomposition

$$\Lambda \rightarrow \bigoplus_{\tilde{\Lambda}} \tilde{\Lambda} , \quad (\text{B.2})$$

where $\tilde{\Lambda} = [\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_N]$ are highest weights of $\mathfrak{u}(N)$ whose rows \tilde{r}_i satisfy

$$r_1 \geq \tilde{r}_1 \geq r_2 \geq \tilde{r}_2 \geq \dots \geq \tilde{r}_N \geq 0 , \quad (\text{B.3})$$

each $\tilde{\Lambda}$ appearing once.

3. In the end, each $\tilde{\Lambda}$ has to be restricted to $\mathfrak{su}(N)$ by removing the last Dynkin label.

Equation (B.3) means that from each row $i = 1, \dots, N$, any number $a_i = 0, \dots, \Lambda_i$ of boxes may be removed, such that the new number of boxes in the i^{th} row becomes

$$\tilde{r}_i = r_i - a_i . \quad (\text{B.4})$$

So the weights $\tilde{\Lambda}$ are labelled by the vectors $\mathbf{a} = (a_1, \dots, a_N)$ and we write $\Lambda(\mathbf{a})$ for the restriction to $\mathfrak{su}(N)$ of the $\tilde{\Lambda}$ labelled by \mathbf{a} . The Dynkin labels of $\Lambda(\mathbf{a})$ are given by

$$\Lambda(\mathbf{a})_i = \tilde{r}_i - \tilde{r}_{i+1} = \Lambda_i - a_i + a_{i+1} \quad (\text{B.5})$$

for $i = 1, \dots, N-1$, and thus the branching rules may be written as

$$\Lambda \rightarrow \bigoplus_{\mathbf{a}} \Lambda(\mathbf{a}) . \quad (\text{B.6})$$

C The ground state analysis

In this appendix we shall show that the coset states (4.3) actually define twisted sector ground states. In particular, we need to show that $\delta h^{(l)}$ in (4.22) is non-negative for all $l = 0, \dots, N-1$. Because the individual twists satisfy $|\alpha_i| \leq \frac{1}{2}$, only the representations with $n = 0$ have a chance of lowering the conformal dimension of the original state. The $\mathfrak{so}(2N)_1$ selection rule implies that $n = \frac{1}{2}$ for the actual fermionic excitations, but $n = 0$ can arise for the bosonic excitations (that come from the same multiplets). Thus we need to analyse (i) whether $n = 0$ is allowed in the fusion with $(0; \mathbf{f}, -(N+1))$ or $(0; \bar{\mathbf{f}}, (N+1))$; and (ii) if so, whether the relevant term in (4.21) is then positive.

The condition that $n = 0$ is possible simply means that $\Lambda_-(\alpha)^{\epsilon(l)}$ is contained in $\Lambda_+(\alpha)$ under the branching rules of $\mathfrak{su}(N+1) \supset \mathfrak{su}(N)$. In the notation of appendix B the original coset state (4.3) corresponds to the choice $\Lambda_+(\alpha) \equiv \Lambda$, and $\Lambda_-(\alpha) \equiv \Lambda(\mathbf{a})$ with

$$\mathbf{a} = \mathbf{a}^{(m)} = (0, \dots, 0, \Lambda_{m+1}, \dots, \Lambda_N) . \quad (\text{C.1})$$

Furthermore, generically the fusion with $(0; \mathbf{f}, -(N+1))$ or $(0; \bar{\mathbf{f}}, (N+1))$ leads to

$$\Lambda(\mathbf{a})^{\epsilon(l)} = \Lambda(\mathbf{a}'), \quad \text{where} \quad a'_j = \begin{cases} a_j & j \neq l+1 \\ a_{l+1} + \epsilon & j = l+1 \end{cases} . \quad (\text{C.2})$$

However, this representation only appears in the above branching rules of the same $\Lambda_+(\alpha) \equiv \Lambda$ if all a'_j satisfy $0 \leq a'_j \leq \Lambda_j$. Thus we see that $n = 0$ is only allowed if

$$\begin{aligned} \text{for } \epsilon = + & \quad a_{l+1} < \Lambda_{l+1} \quad \text{i.e., } l \leq m \\ \text{for } \epsilon = - & \quad 0 < a_{l+1} \quad \text{i.e., } l \geq m+1 . \end{aligned} \quad (\text{C.3})$$

(We are assuming here, for simplicity, that all $\Lambda_j \neq 0$.) But for these values of ϵ and l , it then follows from (4.4) that $-\epsilon \alpha_{l+1} \geq 0$. This therefore shows that $\delta h^{(l)}$ in (4.22) is indeed non-negative.

C.1 Other potential twisted sector ground states

It is also not hard to show that among the ‘light states’, i.e., those that have $n = 0$, the only twisted sector ground states are in fact those described in (4.3). The most general light states are of the form

$$\left(\Lambda; \Lambda(\mathbf{a}), -|\Lambda| + (N+1) \sum_{j=1}^N a_j \right), \quad |\Lambda| = \sum_{j=1}^N j \Lambda_j, \quad (\text{C.4})$$

where $\mathbf{a} = (a_1, \dots, a_N)$, and the a_i take the values $a_i = 0, \dots, \Lambda_i$, $i = 1, \dots, N$. We want to show that among these states, the only ones that are twisted sector ground states, i.e., annihilated by all positive fermionic and bosonic modes, are those for which \mathbf{a} is of the form (C.1). In order to analyse this issue, we determine the analogue of (4.21), which now takes the form

$$\delta h^{(l)} \cong n + \frac{\epsilon}{N+k+1} \left((\Lambda_{l+1} - a_{l+1}) + \sum_{i=l+2}^N \Lambda_i - A \right), \quad A = \sum_{i=1}^N a_i. \quad (\text{C.5})$$

Using (C.2), we have again that $n = 0$ is only allowed for $\epsilon = +$ if $a_{l+1} < \Lambda_{l+1}$, and for $\epsilon = -$ if $a_{l+1} > 0$ — otherwise the representation $\Lambda(\mathbf{a}')$ does not appear in the branching rules of $\mathfrak{su}(N+1) \supset \mathfrak{su}(N)$. It follows that if $0 < a_j < \Lambda_j$, both values $\epsilon = \pm$ allow for $n = 0$ and thus one of the two $\delta h^{(l)}$ will be negative. So for a ground state, each a_j is either $a_j = 0$ or $a_j = \Lambda_j$.

As a last step, we show that in fact $\mathbf{a} = \mathbf{a}^{(m)}$ for some $m = 0, \dots, N$. Requiring (C.5) to be non-negative for all l , we obtain the inequalities

$$\text{if } a_{l+1} = 0 : \quad A \leq \sum_{j=l+1}^N \Lambda_j \quad (\text{C.6})$$

(recall that for $a_{l+1} = 0$, $n = 0$ occurs for $\epsilon = +$) and

$$\text{if } a_{l+1} = \Lambda_{l+1} : \quad \sum_{j=l+2}^N \Lambda_j \leq A \quad (\text{C.7})$$

(since for $a_{l+1} = \Lambda_{l+1}$, $n = 0$ occurs for $\epsilon = -$).

The sequence of partial sums $P_r = \sum_{j=r}^N \Lambda_j$ is strictly decreasing, whereas A takes the same value in all of these inequalities. This implies that the a_j have to be chosen in such a way that $\mathbf{a} = (0, \dots, 0, \Lambda_{m+1}, \dots, \Lambda_N) = \mathbf{a}^{(m)}$. This completes the proof.

References

- [1] M.A. Vasiliev, “Nonlinear equations for symmetric massless higher spin fields in (A)dS(d),” Phys. Lett. B **567** (2003) 139 [arXiv:hep-th/0304049].

- [2] S. Giombi and X. Yin, “The higher spin/vector model duality,” J. Phys. A **46** (2013) 214003 [[arXiv:1208.4036](#) [[hep-th](#)]].
- [3] M.R. Gaberdiel and R. Gopakumar, “Minimal model holography,” J. Phys. A **46** (2013) 214002 [[arXiv:1207.6697](#) [[hep-th](#)]].
- [4] S.F. Prokushkin and M.A. Vasiliev, “Higher spin gauge interactions for massive matter fields in 3d AdS space-time,” Nucl. Phys. B **545** (1999) 385 [[arXiv:hep-th/9806236](#)].
- [5] S.F. Prokushkin and M.A. Vasiliev, “3-d higher spin gauge theories with matter,” [arXiv:hep-th/9812242](#).
- [6] J.R. David, G. Mandal and S.R. Wadia, “Microscopic formulation of black holes in string theory,” Phys. Rept. **369** (2002) 549 [[arXiv:hep-th/0203048](#)].
- [7] M.R. Gaberdiel and R. Gopakumar, “Large $\mathcal{N} = 4$ holography,” JHEP **1309** (2013) 036 [[arXiv:1305.4181](#) [[hep-th](#)]].
- [8] T. Creutzig, Y. Hikida and P.B. Ronne, “Extended higher spin holography and Grassmannian models,” JHEP **1311** (2013) 038 [[arXiv:1306.0466](#) [[hep-th](#)]].
- [9] C. Candu and C. Vollenweider, “On the coset duals of extended higher spin theories,” JHEP **1404** (2014) 145 [[arXiv:1312.5240](#) [[hep-th](#)]].
- [10] M.R. Gaberdiel and C. Peng, “The symmetry of large $N=4$ holography,” [arXiv:1403.2396](#) [[hep-th](#)].
- [11] M. Beccaria, C. Candu and M.R. Gaberdiel, “The large $\mathcal{N} = 4$ superconformal \mathcal{W}_∞ algebra,” [arXiv:1404.1694](#) [[hep-th](#)].
- [12] T. Creutzig, Y. Hikida and P.B. Ronne, “Higher spin AdS_3 holography with extended supersymmetry,” [arXiv:1406.1521](#) [[hep-th](#)].
- [13] M.R. Gaberdiel and R. Gopakumar, “An AdS_3 dual for minimal model CFTs,” Phys. Rev. D **83** (2011) 066007 [[arXiv:1011.2986](#) [[hep-th](#)]].
- [14] M.R. Gaberdiel and R. Gopakumar, “Higher spins & strings,” in preparation.
- [15] Y. Kazama and H. Suzuki, “New $N=2$ superconformal field theories and superstring compactification,” Nucl. Phys. B **321** (1989) 232.
- [16] Y. Kazama and H. Suzuki, “Characterization of $N=2$ superconformal models generated by coset space method,” Phys. Lett. B **216** (1989) 112.
- [17] T. Creutzig, Y. Hikida and P.B. Ronne, “Higher spin AdS_3 supergravity and its dual CFT,” JHEP **1202** (2012) 109 [[arXiv:1111.2139](#) [[hep-th](#)]].
- [18] C. Candu and M.R. Gaberdiel, “Supersymmetric holography on AdS_3 ,” JHEP **1309** (2013) 071 [[arXiv:1203.1939](#) [[hep-th](#)]].
- [19] C. Restuccia, “Limit theories and continuous orbifolds,” [arXiv:1310.6857](#) [[hep-th](#)].

- [20] S. Fredenhagen and C. Restuccia, “The geometry of the limit of $N=2$ minimal models,” J. Phys. A **46** (2013) 045402 [[arXiv:1208.6136 \[hep-th\]](#)].
- [21] M.R. Gaberdiel and P. Suchanek, “Limits of minimal models and continuous orbifolds,” JHEP **1203** (2012) 104 [[arXiv:1112.1708 \[hep-th\]](#)].
- [22] I. Runkel and G.M.T. Watts, “A nonrational CFT with $c = 1$ as a limit of minimal models,” JHEP **0109** (2001) 006 [[arXiv:hep-th/0107118](#)].
- [23] D. Roggenkamp and K. Wendland, “Limits and degenerations of unitary conformal field theories,” Commun. Math. Phys. **251** (2004) 589 [[arXiv:hep-th/0308143](#)].
- [24] S. Fredenhagen, “Boundary conditions in Toda theories and minimal models,” JHEP **1102** (2011) 052 [[arXiv:1012.0485 \[hep-th\]](#)].
- [25] S. Fredenhagen, C. Restuccia and R. Sun, “The limit of $N=(2,2)$ superconformal minimal models,” JHEP **1210** (2012) 141 [[arXiv:1204.0446 \[hep-th\]](#)].
- [26] I.R. Klebanov and A.M. Polyakov, “AdS dual of the critical $O(N)$ vector model,” Phys. Lett. B **550** (2002) 213 [[arXiv:hep-th/0210114](#)].
- [27] E. Sezgin and P. Sundell, “Holography in 4D (super) higher spin theories and a test via cubic scalar couplings,” JHEP **0507** (2005) 044 [[arXiv:hep-th/0305040](#)].
- [28] M. Henneaux, G. Lucena Gomez, J. Park and S.-J. Rey, “Super- $W(\infty)$ asymptotic symmetry of higher-spin AdS_3 supergravity,” JHEP **1206** (2012) 037 [[arXiv:1203.5152 \[hep-th\]](#)].
- [29] K. Hanaki and C. Peng, “Symmetries of holographic super-minimal models,” JHEP **1308** (2013) 030 [[arXiv:1203.5768 \[hep-th\]](#)].
- [30] C. Ahn, “The large N ’t Hooft limit of Kazama-Suzuki model,” JHEP **1208** (2012) 047 [[arXiv:1206.0054 \[hep-th\]](#)].
- [31] C. Candu and M.R. Gaberdiel, “Duality in $N=2$ minimal model holography,” JHEP **1302** (2013) 070 [[arXiv:1207.6646 \[hep-th\]](#)].
- [32] M.R. Gaberdiel and C. Vollenweider, “Minimal model holography for $SO(2N)$,” JHEP **1108** (2011) 104 [[arXiv:1106.2634 \[hep-th\]](#)].
- [33] G. Mussardo, G. Sotkov and M. Stanishkov, “ $N=2$ superconformal minimal models,” Int. J. Mod. Phys. A **4** (1989) 1135.
- [34] G. Mussardo, G. Sotkov and M. Stanishkov, “Fusion rules, four point functions and discrete symmetries of $N = 2$ superconformal models,” Phys. Lett. B **218** (1989) 191.
- [35] M.R. Gaberdiel, “Fusion of twisted representations,” Int. J. Mod. Phys. A **12** (1997) 5183 [[arXiv:hep-th/9607036](#)].
- [36] O. Lunin and S.D. Mathur, “Three point functions for $M(N) / S(N)$ orbifolds with $N=4$ supersymmetry,” Commun. Math. Phys. **227** (2002) 385 [[arXiv:hep-th/0103169](#)].
- [37] H. Weyl, “The theory of groups and quantum mechanics,” Dover Publications, Inc., New York (1931), 2nd ed.

- [38] M.L. Whippman, “Branching rules for simple Lie groups,” J. Math. Phys. **6** (1965) 1534.
- [39] R.C. King, “Branching rules for classical Lie groups using tensor and spinor methods,” J. Phys. A **8** (1975) 429.